Thomson and Rayleigh Scattering

Initial questions: What produces the shapes of emission and absorption lines? What information can we get from them regarding the environment or other conditions?

In this and the next several lectures, we're going to explore in more detail some specific radiative processes. The simplest, and the first we'll do, involves scattering. There are many interesting limits of scattering, but for a start we'll treat frequency-independent nonrelativistic scattering (Thomson scattering).

Suppose a lonely electron has an electromagnetic wave incident on it. The wave has a frequency ω. **Ask class:** what will be the response of the electron to the electric field of the wave? Since the electric field is sinusoidal, the electron will oscillate up and down at the frequency ω. **Ask class:** what will be the result of that acceleration? The electron will radiate. **Ask class:** at what frequency? It will just be at the driving frequency of the radiation, ω. Note also that the radiation is not in the original direction of polarization. Seen from the outside, this means it looks like a photon hits the electron, then scatters off in some new direction. In the limit we're considering, the new frequency is exactly the same as the old frequency. That's Thomson scattering.

If we have a flux (in number per square centimeter per second) of incident particles at some target, and there is some rate of interactions (in number per second), the cross section is just the rate divided by the flux. **Ask class:** in this case, we have an energy flux and a power, so how do we get the cross section? Same idea: power (energy per second) divided by energy flux (energy per area per second). Therefore, to figure out the cross section, we need the radiated power for a given flux.

The electric force from a linearly polarized wave is

$$\mathbf{F} = e\hat{\epsilon}E_0 \sin\omega t \ , \tag{1}$$

where $\hat{\epsilon}$ is the $\mathbf{E}$ field direction. From Newton's second law and the definition of the dipole moment $\mathbf{d} = e\mathbf{r}$, we get

$$\begin{aligned} \ddot{\mathbf{d}} &= (e^2E_0/m)\hat{\epsilon}\sin\omega t \\ \mathbf{d} &= -(e^2E_0/m\omega^2)\hat{\epsilon}\sin\omega t \ . \end{aligned} \tag{2}$$

Note, by the way, a subtle but crucial difference between a single charged particle like the electron we are considering, and a target with multiple electrons that is *polarizable*. For something that is polarizable, an incident electric field shifts the probability distribution of the electron clouds in such a way as to induce a dipole moment that is proportional to the incident electric field. If the constant of proportionality is α_p, this means that the dipole moment is

$$\mathbf{d} = \alpha_p\hat{\epsilon}E_0 \sin\omega t \ . \tag{3}$$

Thus for this situation, the second derivative of the dipole moment is $\ddot{\mathbf{d}} = -\omega^2 \alpha_p \hat{\epsilon} E_0 \sin \omega t$. The factor of ω^2 has important consequences that we will address after we return to Thomson scattering.

Larmor's formula, which gives the total power as $P = 2\ddot{\mathbf{d}}^2/3c^3$, gives the time-averaged power per solid angle:

$$dP/d\Omega = [e^4 E_0^2/(8\pi m^2 c^3)] \sin^2 \Theta \ , \tag{4}$$

where Θ is the angle relative to the initial polarization direction, *not* the propagation direction. The incident flux is $\langle S \rangle = (c/8\pi) E_0^2$, so we can define a differential cross section for scattering into the solid angle $d\Omega$:

$$dP/d\Omega = \langle S \rangle (d\sigma/d\Omega) = (cE_0^2/8\pi)(d\sigma/d\Omega) \ , \tag{5}$$

so

$$d\sigma/d\Omega = (e^4/m^2 c^4) \sin^2 \Theta = r_0^2 \sin^2 \Theta \ , \tag{6}$$

where $r_0 \equiv e^2/mc^2$ is the "classical radius of the electron", $r_0 = 2.8 \times 10^{-13}$ cm. It's the radius that would give an electrostatic energy equal to $m_e c^2$. Integrating over all angles, one finds $\sigma = (8\pi/3) r_0^2$; one can also get this directly from $P = \langle S \rangle \sigma$. For an electron, $\sigma = \sigma_T = 6.65 \times 10^{-25}$ cm^2.

Ask class: from the preceding, how would they expect the Thomson cross section of a proton to compare to that of an electron? The cross section is proportional to $1/m^2$, so the cross section for a proton is tiny, something like 3×10^6 times less than for an electron. That's the reason why, for fully ionized plasmas, interaction of radiation with protons is rarely considered. **Ask class:** qualitatively, why is the cross section of a proton less than that of an electron? It's because the acceleration of a proton is lots less than that of an electron (due to the higher mass), so the radiation power is less as well.

We can now compute the differential cross section for unpolarized light by realizing that it's just the superposition of two orthogonally polarized waves. If we define θ as the angle between the scattered radiation and the original radiation (so that $\theta = \pi/2 - \Theta$), the result is

$$d\sigma/d\Omega = \frac{1}{2} r_0^2 (1 + \cos^2 \theta) \ , \tag{7}$$

and the total cross section is the same as before (as it must be, since an electron at rest has no intrinsic polarization, so it has to react to all linear polarizations in the same way).

We notice a couple of things. First, the radiation is front-back symmetric but not isotropic. There is a tendency for the radiation to be scattered in the direction of motion or directly opposite it. In principle this means that a random walk of scatters is a little different from what it would be with isotropic scattering, but in practice there is not much difference. Second, scattering polarizes radiation (as we've already seen). In particular, if one starts with unpolarized light then the light in direction θ has a degree of polarization

$$\Pi = (1 - \cos^2\theta)/(1 + \cos^2\theta) \ . \tag{8}$$

Light along the original direction is unpolarized. Perpendicular to that direction, it's 100% polarized.

We've assumed that the energy of the photon is much less than the mass-energy of the electron. **Ask class:** if the photon energy were comparable to the electron mass-energy, what difference would they expect that to make? The main difference is that the momentum of the photon is then significant, meaning that the electron has some non-negligible recoil due to the scattering. This leads to Compton scattering, which we discuss in the next lecture.

Now let's get back to what we would have if the target is polarizable. Then $\ddot{\mathbf{d}} \propto \omega^2 \propto \lambda^{-2}$, where λ is the wavelength of the incident radiation. Thus the power scales as $\ddot{\mathbf{d}}^2 \propto \lambda^{-4}$. This is Rayleigh scattering; visible light has a much lower frequency than the natural frequency of molecules in the atmosphere, so blue light scatters a lot more than red light. Thus, blue skies and red sunsets!

We will now consider a generalization of both limits we have discussed. Suppose the target is something with structure (e.g., an atom or molecule). This therefore has a natural frequency. How does this change things? Let's guess, then see if we're right. Suppose that the frequency of the wave is much larger than the natural frequency of the atom. Then the binding of the electron plays little role. **Ask class:** how do they expect the cross section to behave then? It's just the same as if the electron were free, so we should get the Thomson cross section again. Now, suppose that the frequency of the wave is much smaller than the natural frequency. Then, the whole atom essentially rides up and down on the wave. Thus, the acceleration is proportional to the second derivative of the motion. If the motion is like $\cos\omega t$, the acceleration is like $\omega^2\cos\omega t$. **Ask class:** what would this say about the cross section? Remember that the cross section is radiated energy divided by incident flux. Thus, since radiated energy is proportional to a^2, this suggests a form like $\sigma \propto \omega^4$. Let's see how that goes in more detail.

First, let's think about a particle oscillating by itself. If it has a sinusoidal oscillation with frequency ω_0, you might think that the equation of motion would be

$$\ddot{x} + \omega_0^2 x = 0 \ . \tag{9}$$

Ask class: is there any other contribution to the motion? Remember that an accelerated particle emits radiation, so we have to take radiation reaction into account. The averaged force is $-\tau x^{(3)}$, where $x^{(3)}$ is the third time derivative. However, we can simplify by realizing that radiation reaction is a small perturbation (unless the oscillation period is $\sim 10^{-23}$ s or less!), so we'll say that $x^{(3)} \approx -\omega_0^2 \dot{x}$. Then we find

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0 \; . \tag{10}$$

Note that the middle term has an odd number of time derivatives. That means that it is not time reversible. That's one of the signatures of damping, which is also not time reversible.

Let's add a driving force. Suppose that an external time-varying electric field is added, which contributes a force $eE_0 \cos \omega t$. This is what an incident photon would do. A common trick here is to use $\exp(i\omega t)$ instead of $\cos \omega t$, and remember to take the real part later. We get

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = (eE_0/m) \exp(i\omega t) \; . \tag{11}$$

When confronted with differential equations of this type, a good approach is to try a solution of the form $x = A \exp(\alpha t)$, then solve for A and α after substitution. When you do this, you find

$$x = |x_0| \exp[i(\omega t + \delta)] \tag{12}$$

where

$$\begin{aligned} x_0 &= -(eE_0/m)\left(\omega^2 - \omega_0^2 - i\omega_0^3 \tau\right)^{-1} \\ \delta &= \tan^{-1}[\omega^3 \tau / (\omega^2 - \omega_0^2)] \; . \end{aligned} \tag{13}$$

Looking past all the various factors, what this basically says is that the electron *does* oscillate at the driving frequency ω, but delayed (or advanced) by the phase shift δ. We can now figure out the power radiated by getting the time average using the Larmor formula and taking the real part:

$$P = e^2 |x_0|^2 \omega^4 / (3c^3) = \frac{e^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3 \tau)^2} \; . \tag{14}$$

The time-averaged flux is $\langle S \rangle = (c/8\pi) E_0^2$, so the cross section is the power over the flux or

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3 \tau)^2} \; . \tag{15}$$

Note that since τ is so small, $\tau \omega_0 \ll 1$.

Ask class: what's one limit we could explore? We could look at the limit $\omega \gg \omega_0$. **Ask class:** what is the answer in that limit? It's just σ_T. This corresponds to our expectations. **Ask class:** what about when $\omega \ll \omega_0$? Then $\sigma(\omega) = \sigma_T (\omega/\omega_0)^4$, which is what we got before for Rayleigh scattering.

Now let's explore another limit in some more detail. Suppose that the frequency ω of the photon is close to the natural (or resonant) frequency ω_0 of the atom/molecule. The only subtlety here is dealing with the $(\omega^2 - \omega_0^2)^2$ term in the bottom. A standard method is to retain factors such as $\omega-\omega_0$, but otherwise set ω to ω_0. Then we get

$$\begin{aligned}\sigma(\omega) &= \frac{\pi\sigma_T}{2\tau}\frac{\omega_0^2\tau/2\pi}{(\omega-\omega_0)^2+(\omega_0^2\tau/2)^2} \\ &= \frac{\pi\sigma_T}{2\tau}\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2} \\ &= \frac{2\pi^2e^2}{mc}\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}\end{aligned} \tag{16}$$

where $\Gamma=\omega_0^2\tau$ is the "natural width" of the resonance and we have used $\tau=2e^2/3mc^3$ from Lecture 11 and $\sigma_T=(8\pi/3)e^4/m^2c^4$ from earlier in this lecture. This profile is known as a Lorentz profile for a spectral line. Incidentally, near the resonance the shape of the cross section is the same as the shape of a line from free oscillations of the atom. That's not accidental; one could excite the free oscillations by having a pulse of radiation near ω_0 hit the atom, then ring down freely.

This whole approach has interest in part because it provides the only classical description of a spectral line. Often when one gets the correct quantum result, one refers the cross section to the classical result to get a ratio. For example, let's integrate that line over angular frequency ω or the cycle frequency ν:

$$\begin{aligned}\int_0^\infty \sigma(\omega)\,d\omega &= 2\pi^2e^2/(mc) \\ \int_0^\infty \sigma(\nu)\,d\nu &= \pi e^2/(mc)\ .\end{aligned} \tag{17}$$

But wait, didn't we cheat? We found that at high frequencies the cross section approaches a constant value (the cross section σ_T of Thomson scattering). This would mean that the total integral is infinite! Oops. This type of situation is encountered in a number of places in physics; the formal integral is infinite, but you know that physically it must be finite. One way to deal with this is to impose a cutoff on the integral, and (1) show that this is justified physically, and (2) show that the precise value of the cutoff doesn't make much difference. **Ask class:** in this case, what cutoff can we make and how can we justify it? We know that the radiation reaction formula is only valid if $\omega\tau\ll 1$, so if ω gets too large, all bets are off. Now, does the frequency at which we cut off the integral make a difference? The integral of just the Thomson cross section is $\int_0^{\omega_{\rm max}}\sigma_T\,d\omega=\sigma_T\omega_{\rm max}$. We know that $\omega_{\rm max}\ll 1/\tau$, so $\sigma_T\omega_{\rm max}\ll\sigma_T/\tau=4\pi e^2/mc$, so the precise value can't matter much. Therefore, our little "swindle" above was okay.

The preceding was a classical attempt to model a spectral line of an atom. In the quantum theory one does things differently, but it is still convenient to define the "oscillator strength" $f_{nn'}$ of a transition from state n to n' by

$$\int_0^\infty \sigma(\nu)\,d\nu=\frac{\pi e^2}{mc}f_{nn'}\ . \tag{18}$$

Recommended Rybicki and Lightman problem: 3.5